

## FREE $\alpha$ -EXTENSIONS OF AN ARCHIMEDEAN VECTOR LATTICE AND THEIR TOPOLOGICAL DUALS

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**ABSTRACT.** Arch denotes the category of Archimedean vector lattices with vector lattice homomorphisms, and  $\alpha$  denotes an uncountable cardinal number or the symbol  $\infty$ . Arch( $\alpha$ ) denotes the category of Arch objects with  $\alpha$ -complete Arch morphisms.

In this paper we construct, for each  $L \in |\text{Arch}|$ ,  $\alpha$ -complete extensions  $L'$  of  $L$  that lift Arch morphisms from  $L$  to  $\alpha$ -complete Arch morphisms from  $L'$ . Specifically, we construct the *free  $\alpha$ -extension* and the *free  $\alpha$ -regular extension* of an Arch object  $L$ . By virtue of the latter, the full subcategory of  $\alpha$ -complete objects, in Arch( $\alpha$ ), is epireflective. The proofs work in Boolean algebras and recover the results obtained in [K, Y, and S]. Our proofs are different and, it can be argued, more natural.

$\mathcal{W}$  denotes the category of Arch objects with distinguished weak unit and Arch morphisms that preserve units. We exploit a certain contravariant functor  $Y : \mathcal{W} \rightarrow \text{Comp}$  (the so-called Yosida functor, analogous to the Stone-space functor) from  $\mathcal{W}$  to the category of compact Hausdorff spaces with continuous functions, to convert algebraic results in  $\mathcal{W}$  to topological results in the topological category  $\alpha\text{-SpFi}$ . Specifically, we show that the Yosida “dual” of the  $\mathcal{W}$ -free  $\alpha$ -regular extension of  $C(X)$  is the  $\alpha$ -disconnected  $\alpha\text{-SpFi}$  monoreflection of the compact space  $X$ , thereby showing that the full subcategory of  $\alpha$ -disconnected spaces, in  $\alpha\text{-SpFi}$ , is monoreflective.

### 1. INTRODUCTION

$\alpha$  denotes an uncountable cardinal number or the symbol  $\infty$ . The meaning of  $\alpha = \infty$  will be clear from the context. When we write  $\alpha < \infty$  or  $|A| < \infty$ , where  $A$  is a set, we mean that  $\alpha$  or  $|A|$  is an arbitrary cardinal number.

Arch denotes the category of Archimedean vector lattices with vector lattice homomorphisms. An element  $u \in L \in |\text{Arch}|$  is called a *weak unit* if the band (complete ideal) generated by  $u$  is all of  $L$  [LZ, dJvR].  $u$  is called a *strong unit* if the principal ideal generated by  $u$  is all of  $L$ .  $\mathcal{W}$  ( $\mathcal{S}$ ) denote the category of Arch objects with distinguished weak (strong) unit and unit preserving Arch

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morphisms. Obviously a strong unit is a weak unit, so  $\mathcal{S}$  is a subcategory of  $\mathcal{W}$ .

An Arch morphism  $\varphi : L \rightarrow M$  is called  $\alpha$ -complete if, for  $A \subset L$  with  $|A| < \alpha$ , we have that  $\varphi(\bigvee^L A) = \bigvee^M \varphi[A]$  whenever  $\bigvee^L A$  exists in  $L$ .  $\text{Arch}(\alpha)$ ,  $\mathcal{W}(\alpha)$ , and  $\mathcal{S}(\alpha)$  denote the categories of Arch,  $\mathcal{W}$ , and  $\mathcal{S}$  objects with  $\alpha$ -complete morphisms respectively.

An  $L \in |\text{Arch}|$  is called  $\alpha$ -complete  $\bigvee^L A$  exists in  $L$  for all bounded above  $A \subset L$  with  $|A| < \alpha$ .

Recall  $L$  is called *Dedekind complete* if every subset of  $L$  that is bounded above has a supremum in  $L$ ; thus  $L$  is  $\infty$ -complete if and only if  $L$  is Dedekind complete.

$L \subseteq M$  denotes that  $L$  is an Arch subspace of  $M$  (i.e.,  $L$  is a vector lattice subspace of  $M$ ), while  $L \subseteq^\alpha M$  denotes that, in addition to  $L \subseteq M$ , the inclusion of  $L$  into  $M$  is an  $\alpha$ -complete Arch morphism. As usual, we reserve  $\subset$  for ordinary set inclusion.

The next two definitions, as do the definitions of an  $\alpha$ -complete morphism and an  $\alpha$ -complete object, make sense in  $\mathcal{W}$ ,  $\mathcal{S}$ , and Boolean algebras ( $\equiv \mathcal{BA}$ ), as well as in Arch.

For  $L \subseteq M$ , we say that  $L$  is  $\alpha$ -full subspace in, or of,  $M$  if  $b = \bigvee^M \{A : A \subset L, |A| < \alpha\}$  implies that  $b \in L$  (i.e.,  $b = \bigvee^L A$ ). We call an Arch embedding  $\varphi : L \hookrightarrow M$   $\alpha$ -full if  $\varphi[L]$  is  $\alpha$ -full in  $M$ .

What we call an  $\alpha$ -full subspace of  $M$ , Sikorski (in §23 of [S] for Boolean algebras) calls an  $\alpha$ -subalgebra of  $M$ .

For  $L \subseteq M$ , we say that  $L$   $\alpha$ -generates  $M$  if  $M$  is the smallest  $\alpha$ -full subspace of  $M$  that contains  $L$ . That is, if  $L'$  is  $\alpha$ -full in  $M$  and  $L \subseteq L'$ , then  $L' = M$ .

Let  $L$  and  $M$  be Arch objects and let  $\varphi : L \hookrightarrow M$  be an Arch morphism with  $M$   $\alpha$ -complete. We say the pair  $(\varphi, M)$  is an Arch *free  $\alpha$ -extension* of  $L$  if  $\varphi[L]$   $\alpha$ -generates  $M$ , and each Arch morphism from  $L$  into an  $\alpha$ -complete Arch object  $N$  can be extended to an  $\alpha$ -complete Arch morphism from  $M$  into  $N$ . We say the pair  $(\varphi, M)$  is an Arch *free  $\alpha$ -regular extension* of  $L$  if  $\varphi$  is an Arch  $\alpha$ -complete embedding,  $\varphi[L]$   $\alpha$ -generates  $M$ , and each  $\alpha$ -complete Arch morphism from  $L$  into an  $\alpha$ -complete Arch object  $N$  can be extended to an  $\alpha$ -complete Arch morphism from  $M$  into  $N$ .

Analogously, one obtains the definitions of a  $\mathcal{W}$ ,  $\mathcal{S}$ , or  $\mathcal{BA}$  free  $\alpha$ -regular extension and a  $\mathcal{W}$ ,  $\mathcal{S}$ , or  $\mathcal{BA}$  free  $\alpha$ -extension by replacing the symbol Arch with  $\mathcal{W}$ ,  $\mathcal{S}$ , or  $\mathcal{BA}$  in the above.

In §3, we indicate that, for each  $L$ , there is essentially, one free  $\alpha$ -regular extension, and one free  $\alpha$ -extension, of  $L$ .

Henceforth,  $L$ ,  $M$ , and  $N$  denote Arch objects, and maps between them are always considered to be Arch morphisms unless otherwise stated.

For  $\alpha < \infty$ , Yaqub shows in [Y] that every Boolean algebra has a free  $\alpha$ -extension. However, in [Ha], Hales shows that the free Boolean algebra on  $\omega$  generators does not have a free  $\infty$ -extension.

The existence of the free  $\alpha$ -regular extension of a Boolean algebra (for  $\alpha < \infty$ ) was first proved by Kerstan in [K], and subsequently, independently by Sikorski and Yaqub in [S<sub>2</sub> and Y] respectively. In [S], Sikorski expands on the idea of free  $\alpha$ -regular extensions of Boolean algebras and discusses what

he calls  $(J, M, m)$ -extensions. Also, since in  $\mathcal{BA}$ , the injective objects are exactly the  $\infty$ -complete Boolean algebras [S, 33.1], the free  $\infty$ -regular extension of a Boolean algebra  $B$  is the Dedekind completion of  $B$  [S]. Moreover, even though there are no injectives in Arch [Co], the  $\infty$ -regular extension of an  $L \in |\text{Arch}|$  is the Arch Dedekind completion of  $L$  [BH<sub>1</sub>, M<sub>1</sub>].

What we do here in §3, for  $\alpha < \infty$ , is construct the Arch free  $\alpha$ -extension and the Arch free  $\alpha$ -regular extension of an Arch object  $L$ . The proofs work in  $\mathcal{W}$  and  $\mathcal{S}$ , as well as in  $\mathcal{BA}$ , and recover the results obtained in [K, Y, and S]. Our proofs are different, and, it can be argued, more natural.

In §4 here we exploit a certain contravariant functor  $Y : \mathcal{W} \rightarrow \text{Comp}$  (the so-called Yosida functor, analogous to the Stone-space functor) from  $\mathcal{W}$  to the category of compact Hausdorff spaces with continuous functions, to convert algebraic results in  $\mathcal{W}$  to topological results in the topological category  $\alpha\text{-SpFi}$ . Specifically, we show that the Yosida “dual” of the  $\mathcal{W}$ -free  $\alpha$ -regular extension of  $C(X)$  is the  $\alpha$ -disconnected  $\alpha\text{-SpFi}$  *monocoreflection* of the compact space  $X$ , thereby showing that the full subcategory of  $\alpha$ -disconnected spaces, in  $\alpha\text{-SpFi}$ , is *monocoreflective*.

We begin a discussion about the Yosida functor and the category  $\alpha\text{-SpFi}$ .

## 2. THE YOSIDA FUNCTOR AND THE CATEGORY $\alpha\text{-SpFi}$

$X, Y$ , and  $Z$  denote compact Hausdorff spaces;  $f, g$ , and  $h$  denote continuous functions; and  $\text{Comp}$  denotes the category of compact Hausdorff spaces with continuous functions.

We review the Yosida representation theory. For each  $L \in |\mathcal{W}|$  there is an associated compact Hausdorff space,  $Y(L)$ , called the *Yosida space* of  $L$ . See [LZ, BKW, and HeR]. Also for each  $\varphi : L \rightarrow M$ , there is associated continuous function  $Y(\varphi) : Y(M) \rightarrow Y(L)$ . See [HR<sub>1</sub>]. It turns out that  $Y$  is a faithful (contravariant) functor from  $\mathcal{W}$  to  $\text{Comp}$ . The functor  $Y$  works very much like the Stone functor from Boolean algebras to Boolean spaces.  $Y(L)$ , like the Stone space of a Boolean algebra, is a maximal ideal space. The elements of  $Y(L)$  are ideals of  $L$  that are maximal for the property of not containing the weak unit. If the weak unit is a strong unit then these ideals are the actual maximal ideals of  $L$ . The topology on this space is the hull-kernel topology. In fact, if we view a Boolean algebra,  $B$ , as a  $\mathcal{W}$  object (i.e., if  $L(S(B))$  is the locally constant real-valued functions on the Stone space  $S(B)$ , then  $L(S(B_1)) \cong L(S(B_2))$  iff  $B_1 \cong B_2$ ), the Yosida functor can be thought of as an extension of the Stone functor.

The following paragraph comes from [BH<sub>2</sub>].

The archetypal  $\mathcal{W}$  object is  $C(X)$  (the ring of continuous real-valued functions on  $X$  with the pointwise sup and inf). The weak unit of  $C(X)$  will always be taken to be the constant function  $\mathbf{1}$ . Note that the weak unit  $\mathbf{1}$  is indeed a strong unit and  $(C(X), \mathbf{1})$  is an  $\mathcal{S}$  object. Let  $D(X)$  be the set of extended real-valued continuous functions,  $f : X \rightarrow [-\infty, +\infty]$ , for which  $f^{-1}(\mathbf{R})$  is dense in  $X$ . In the pointwise order,  $D(X)$  is a lattice, but usually fails to be a vector space. For  $f, g, h \in D(X)$ , we say “ $f + g = h$  in  $D(X)$ ” if  $f(x) + g(x) = h(x)$  when  $x \in f^{-1}(\mathbf{R}) \cap g^{-1}(\mathbf{R}) \cap h^{-1}(\mathbf{R})$  (which is a dense set in  $X$ ). It may well happen that, for particular  $f, g \in D(X)$ , there is no  $h \in D(X)$  with  $f + g = h$  in  $D(X)$  (e.g., take  $X = [-\infty, +\infty]$ ,  $f$  the obvious

extension of  $x + \sin x$ , and  $g$  the extension of  $-x$ ). However, it may well happen that a subset  $L \subset D(X)$  has the property that for all  $f, g \in L$  there is an  $h \in D(X)$  with  $f + g = h$  in  $D(X)$ ; if  $L$  is also a vector lattice under the pointwise operation in  $D(X)$  and the constant function,  $\mathbf{1}$ , is in  $L$ , then we say “ $(L, \mathbf{1})$  (or just  $L$ ) is a  $\mathscr{W}$  object in  $D(X)$ ”, (e.g.,  $C(X)$  is a  $\mathscr{W}$  object, in  $D(X)$ ). If  $X$  has the property that each dense cozero set is  $C^*$ -embedded [GJ], then  $X$  is called  $\omega_1$ -quasi- $F$  (or just quasi- $F$ ) [DHH, HVW, BHN, M<sub>2</sub>]. If  $X$  is  $\omega_1$ -quasi- $F$ , then  $(D(X), \mathbf{1}) \in |\mathscr{W}|$ . See [HJ].

**2.1 Theorem** (see [BKW, HR1]). (a) *There is a  $\mathscr{W}$  isomorphism,  $\wedge : L \rightarrow \hat{L} \subset D(Y(L))$ , onto a  $\mathscr{W}$  object,  $\hat{L}$  in  $D(Y(L))$ , with  $\hat{w}_L = \mathbf{1}$ , and  $\hat{L}$  separates the points of  $Y(L)$ .*

(b) *If  $L'$  is a  $\mathscr{W}$  object in  $D(X)$  which separates the points of  $X$ , and for  $a \in L$ , if  $a \mapsto a'$  is a  $\mathscr{W}$  isomorphism from  $L$  to  $L'$ , then there is a homeomorphism  $f : X \rightarrow Y(L)$  such that  $a' = \hat{a} \circ f$  for all  $a \in L$ .*

2.1(b) is used to recognize Yosida representations.

**2.2 Corollary.**  $Y(C(X)) = X$ .

*Proof.* In  $D(X)$ ,  $C(X)$  satisfies 2.1(b).

**2.3 Theorem** [HR1]. *Let  $\varphi_i : L \rightarrow M$  for  $i = 1, 2$ .*

(a) *There is a unique continuous function,  $Y(\varphi_1) : Y(M) \rightarrow Y(L)$ , such that  $\varphi(a)^\wedge = \hat{a} \circ Y(\varphi_1)$  for all  $a \in L$ .*

(b)  *$Y$  is a faithful functor, i.e., if  $\varphi_1 \neq \varphi_2$ , then  $Y(\varphi_1) \neq Y(\varphi_2)$ .*

(c)  *$\varphi_1$  is one-to-one if and only if  $Y(\varphi_1)$  is onto, and if  $\varphi_1$  is onto, then  $Y(\varphi_1)$  is one-to-one.*

(d) *Let  $\gamma : C(X) \rightarrow M$  and  $M \in |\mathscr{S}|$ . Then  $Y(\gamma) : Y(M) \rightarrow X$  is one-to-one if and only if  $\gamma$  is onto.*

*Henceforth,  $L$  and  $\hat{L}$  are identified.*

*Thus we will consider  $a \in L$  as an extended real-valued function on  $Y(L)$ .*

The next proposition is straightforward.

**2.4 Proposition.** *Let  $f : X \rightarrow Y$ . Define  $f' : C(Y) \rightarrow C(X)$  by  $f'(g) = g \circ f$  for  $g \in C(Y)$  [GJ, 10.2]. Then  $f'$  is a  $\mathscr{W}$  morphism and  $Y(f') = f$ .*

**2.5 Theorem** (Banach-Stone). *A function  $f : X \rightarrow Y$  is a homeomorphism if and only if  $f' : C(Y) \rightarrow C(X)$  is an isomorphism in  $\mathscr{W}$ .*

*Proof.*  $Y(f') = f$ . Apply 2.3(c) and (d).

We begin a discussion of the topological category  $\alpha\text{-}SpFi$ . See [BHM, BHN, M<sub>1</sub>, M<sub>2</sub>, and BH<sub>3</sub>].

Let  $\text{Coz}(X) = \{f^{-1}(\mathbf{R} \setminus \{0\}) : f \in C(X)\}$ . A subset  $V \subset X$  is said to be an  $\alpha$ -cozero set if

$$V = \bigcup \{U_i : i \in I, |I| < \alpha, U_i \in \text{Coz}(X)\}.$$

Note that an  $\omega_1$ -cozero set is a cozero set. Recall that by “ $|I| < \infty$ ” we mean that “ $|I|$  is unrestricted,” so that every open set is an  $\infty$ -cozero set. We denote the collection of  $\alpha$ -cozero sets of  $X$  by  $\text{Coz}_\alpha(X)$ . Let  $G_\alpha X$  denote the filter base of dense members of  $\text{Coz}_\alpha(X)$ .

A continuous function  $f : X \rightarrow Y$  is called an  $\alpha$ -*SpFi* morphism if  $f^{-1}(G) \in G_\alpha X$  whenever  $G \in G_\alpha Y$ . We thus have a topological category, denoted  $\alpha$ -*SpFi*, which consists of compact Hausdorff spaces and  $\alpha$ -*SpFi* morphisms.

Below (2.6) is a cardinal generalization of 4.2(c) of [BH<sub>1</sub>]. See also [M<sub>2</sub>]. It tells us that the Yosida functor converts an  $\alpha$ -complete  $\mathscr{W}$  morphism into an  $\alpha$ -*SpFi* morphism, and, that every  $\alpha$ -*SpFi* morphism arises from an  $\alpha$ -complete  $\mathscr{W}$  morphism.

**2.6 Lemma.**  $\varphi : L \rightarrow M$  is  $\alpha$ -complete if and only if  $Y(\varphi) : Y(M) \rightarrow Y(L)$  is an  $\alpha$ -*SpFi* morphism. Moreover, a function  $f : X \rightarrow Y$  is an  $\alpha$ -*SpFi* morphism if and only if  $f'$  (2.5) is  $\alpha$ -complete.

From this result it is routine to see that  $Y$  restricted to  $\mathscr{W}(\alpha)$  is a functor to  $\alpha$ -*SpFi*.

Moreover,  $Y$  take  $\mathscr{W}(\alpha)$  epics to  $\alpha$ -*SpFi* monics. Recall that in a category, a morphism  $e$  is called epic if  $f_1 \circ e = f_2 \circ e$  implies that  $f_1 = f_2$ , and a morphism  $m$  is called monic if  $m \circ f_1 = m \circ f_2$  implies that  $f_1 = f_2$ . We have

**2.7 Lemma** [M<sub>2</sub>].  $\varphi : L \rightarrow M$  is epic in  $\mathscr{W}(\alpha)$  if and only if  $Y(\varphi) : Y(M) \rightarrow Y(L)$  is monic in  $\alpha$ -*SpFi*.

Note that epics in  $\mathscr{W}(\alpha)$  are not always surjective, nor are the monics in  $\alpha$ -*SpFi* always injective [BHM].

A space  $X$  is said to be  $\alpha$ -disconnected if the closure of every  $\alpha$ -cozero is open. It is obvious that the notions of  $\infty$ -disconnected and extremally disconnected (and  $\omega_1$ -disconnected and basically disconnected) are equivalent.

As the next two statements indicate, the topological concept of  $\alpha$ -disconnected has an algebraic counterpart. The first is a cardinal generalization of (3.3) ((a)  $\Rightarrow$  (d)) of [BH<sub>1</sub>]. See also [M<sub>1</sub>].

**2.8 Lemma.** Let  $L \in |\mathscr{W}|$ . If  $L$  is  $\alpha$ -complete, then  $Y(L)$  is  $\alpha$ -disconnected.

The converse of 2.8 is not true [BH<sub>1</sub>, M<sub>1</sub>, M<sub>2</sub>]. However, if we only consider  $\mathscr{W}$  objects of the form  $C(X)$ , for compact  $X$ , we get the following well-known result:

**2.9 Theorem** (Stone-Nakano).  $C(X)$  is  $\alpha$ -complete if and only if  $X$  is  $\alpha$ -disconnected.

Note that one direction of 2.9 follows from 2.8 and the fact that  $Y(C(X)) = X$  (2.2).

### 3. FREE $\alpha$ -EXTENSIONS

**3.1 Proposition** (see [M<sub>2</sub>]). Let  $\varphi : L \rightarrow M$ . The following are equivalent.

- (a)  $\varphi$  is  $\alpha$ -complete.
- (b) There is a  $c \in L$  such that whenever  $B \subset L$ ,  $|B| < \alpha$ , and  $c = \bigvee^L B$ , then  $\varphi(c) = \bigvee^M \varphi[B]$ .
- (c) For  $A \subset L$  with  $|A| < \alpha$ , we have that  $\varphi(\bigwedge^L A) = \bigwedge^M \varphi[A]$  whenever  $\bigwedge^L A$  exists in  $L$ .

Sometimes, as the next proposition indicates, the first factor of an  $\alpha$ -complete map is also  $\alpha$ -complete. As usual,  $\gamma : M \hookrightarrow N$  denotes that  $\gamma$  is injective.

**3.2 Proposition.** *Let  $\varphi: L \rightarrow M$  and  $\gamma: M \hookrightarrow N$ . If  $\gamma \circ \varphi$  is  $\alpha$ -complete, then  $\varphi$  is  $\alpha$ -complete.*

*Proof.* Let  $c \in L$  and suppose  $c = \bigvee^L A$  where  $A \subset L$  and  $|A| < \alpha$ . We claim that  $\varphi(c) = \bigvee^M \varphi[A]$ . Suppose not. Then there is a  $b \in M$  such that  $\varphi(c) > b > \varphi(a)$  for all  $a \in A$  ( $>$  means strictly greater than). Because  $\gamma$  is injective we have  $\gamma \circ \varphi(c) > \gamma(b) > \gamma \circ \varphi(a)$  for all  $a \in A$ . But this contradicts the assumption that  $\gamma \circ \varphi$  is  $\alpha$ -complete. For then  $\gamma \circ \varphi(c) = \bigvee^N \gamma \circ \varphi[A]$ .

Recall from the introduction the definition of an  $\alpha$ -full subspace.

Note, for  $L \subseteq M$ , it is possible for  $L$  to be  $\alpha$ -full, but not  $\alpha$ -completely embedded, in  $M$ .

**3.3 Example.** Let  $X$  be a non- $\omega_1$ -disconnected Boolean space. Then there is a cozero set  $U$  such that  $\overline{U}$  is not clopen. Let  $L = \{f \in C(X) : f[X \setminus U] = r \text{ for some } r \in \mathbb{R}\}$ . Clearly,  $L$  is  $\infty$ -full in  $C(X)$ , but  $L$  is not even  $\omega_1$ -completely embedded in  $C(X)$ : Since there are clopen sets  $\{V_n : n \in \mathbb{N}\}$  and  $C$  such that  $U = \bigcup_n V_n$  and  $U \subseteq C$ , it is obvious that the sup of  $\{\chi_{V_n}\}$  in  $C(X)$  is not  $\mathbf{1}$  ( $\chi_c \geq \chi_{V_n}$  for all  $n \in \mathbb{N}$ ). However,  $\bigvee_n \chi_{V_n} = \mathbf{1}$  because, if  $h \in L$  and  $h \geq \chi_{V_n}$  for all  $n \in \mathbb{N}$ , then  $h(x) \geq 1$  for all  $x \in U$ . Therefore  $h(x) \geq 1$  for all  $x \in \overline{U}$ , but then, since  $X \setminus U \cap \overline{U} \neq \emptyset$ , it follows that  $h \geq \mathbf{1}$ .

However, if  $M$  is  $\alpha$ -complete, then the  $\alpha$ -full subspaces of  $M$  are easy to identify. See [S, §23]. We have

**3.4 Proposition.** *Let  $\varphi: L \hookrightarrow M$  be an embedding and suppose  $M$  is  $\alpha$ -complete. Then  $\varphi[L]$  is  $\alpha$ -full in  $M$  if and only if both  $L$  and  $\varphi$  are  $\alpha$ -complete.*

*Proof.* Verification is straightforward.

**3.5 Proposition.** *Let  $L \subseteq M \subseteq N$ . If  $L$  is  $\alpha$ -full in  $M$  and  $M$  is  $\alpha$ -full in  $N$ , then  $L$  is  $\alpha$ -full in  $N$ .*

*Proof.* Let  $A \subset L$ ,  $|A| < \alpha$ , and  $a = \bigvee^N A$ . Since  $M$  is  $\alpha$ -full in  $N$ , we have that  $a \in M$  and  $\bigvee^M A = \bigvee^N A$ . Hence, since  $L$  is  $\alpha$ -full in  $M$ , it follows that  $\bigvee^L A = \bigvee^M A = \bigvee^N A$ .

Let  $S \subset L$ . The smallest subspace of  $L$  that contains  $S$  is the intersection of all the subspaces of  $L$  that contain  $S$ . We denote this subspace by  $\langle S \rangle^L$ , and we say that  $S$  generates  $\langle S \rangle^L$  in  $L$ . Moreover,  $\langle S \rangle^L$  is the smallest subset of  $L$  that contains  $S$  and is closed under the finite vector lattice operations.

Now for given  $\alpha$ , one may ask if there is a smallest  $\alpha$ -complete subspace of  $L$  that contains  $S$ ? In general, this question does not always make sense because there may not be any  $\alpha$ -complete subspace of  $L$  that contain  $S$  (e.g.,  $C([0, 1])$  contains no  $\omega_1$ -complete subspaces). However, there are always  $\alpha$ -full subspaces of  $L$  that contain  $S$ .

Recall, for  $L \subseteq M$ , we say that  $L$   $\alpha$ -generates  $M$  if  $M$  is the smallest  $\alpha$ -full subspace of  $M$  that contains  $L$ . That is, if  $L'$  is  $\alpha$ -full in  $M$  and  $L \subseteq L'$ , then  $L' = M$ . For  $S \subseteq M$  we say  $S$   $\alpha$ -generates  $M$  if  $\langle S \rangle$   $\alpha$ -generates  $M$ .

Clearly, an arbitrary intersection of  $\alpha$ -full subspaces of  $M$  is an  $\alpha$ -full subspace of  $M$ . Therefore, for  $L \subseteq M$ , the subspace of  $M$  which  $L$   $\alpha$ -generates, denoted  $\langle L \rangle_\alpha^M$ , is the intersection of all the  $\alpha$ -full subspaces of  $M$  that contain

$L$ . However, this “outside in” description does not provide much information about  $\langle L \rangle_\alpha^M$ . Is the inclusion of  $L$  into  $\langle L \rangle_\alpha^M$  epic in some sense? How big is  $\langle L \rangle_\alpha^M$ ? To answer these and other questions about  $\langle L \rangle_\alpha^M$  we use an “inside out” construction of  $\langle L \rangle_\alpha^M$ .

In what follows below, we may assume, without loss of generality, that  $\alpha$  is a regular cardinal because:  $\alpha^+$  is always a regular cardinal, and the  $\alpha$ -completeness properties of objects and morphisms are equivalent to their respective  $\alpha^+$ -completeness properties when  $\alpha$  is a singular cardinal.

**3.6 Definition.** Let  $L \subseteq M$ . Fix an  $\alpha < \infty$  and define

$$S_\alpha^M(L) = \left\{ \bigvee^M A : A \subset L, |A| < \alpha, \bigvee^M A \text{ exists} \right\}.$$

For ordinals  $\xi < \alpha$  we recursively define  $L(\xi)^M$  (omitting  $^M$  when the context is clear) by setting:

$$L(0) = L,$$

$$L(\xi) = \langle S_\alpha(L(\xi - 1)) \rangle \text{ if } \xi \text{ is not a limit ordinal,}$$

$$L(\xi) = \bigcup_{\sigma < \xi} L(\sigma) \text{ if } \xi \text{ is a limit ordinal.}$$

Finally, we define  $\langle L \rangle_\alpha^M = \bigcup_{\xi < \alpha} L(\xi)$ .

Note, we also omit the superscript  $^M$  in  $\langle L \rangle_\alpha^M$  whenever the context is clear.

**3.7 Lemma.** Let  $L \subseteq M$  and  $\alpha < \infty$ .

(a)  $\langle L \rangle_\alpha$  is a subspace of  $M$ .

(b) If two  $\alpha$ -complete morphisms from  $\langle L \rangle_\alpha$  agree on  $L$ , they are equal. One might say the embedding of  $L$  into  $\langle L \rangle_\alpha$  is “epic for  $\alpha$ -complete morphisms” [H].

(c) If  $L \subseteq^\alpha M$ , then the embedding of  $L$  into  $\langle L \rangle_\alpha$  is epic in  $\text{Arch}(\alpha)$ .

(d)  $|\langle L \rangle_\alpha| \leq |L|^\alpha$ .

(e) If  $L \subseteq M \subseteq N$  and  $M$  is  $\alpha$ -full in  $N$ , then  $\langle L \rangle_\alpha \subseteq M$ .

(f)  $\langle L \rangle_\alpha$  is  $\alpha$ -full in  $M$ , and  $L$   $\alpha$ -generates  $\langle L \rangle_\alpha$ .

(g) If  $M$  is  $\alpha$ -complete, then  $\langle L \rangle_\alpha$  is  $\alpha$ -complete and  $\langle L \rangle_\alpha \subseteq^\alpha M$ .

*Proof.* (a) Clear.

(b) Let  $\gamma_i : \langle L \rangle_\alpha \rightarrow N$  be  $\alpha$ -complete with  $i = 1, 2$ , and suppose  $\gamma_1|_L = \gamma_2|_L$ . We claim  $\gamma_1|_{L(\xi)} = \gamma_2|_{L(\xi)}$  for all  $\xi < \alpha$ , from whence, (b) will follow. We proceed by transfinite induction. The claim is true for  $L(0) = L$ . Suppose it is true for all ordinals  $\sigma < \xi$ . If  $\xi$  is a limit ordinal, it is clear from the definition of  $L(\xi)$  that  $\gamma_1|_{L(\xi)} = \gamma_2|_{L(\xi)}$ . If  $\xi$  is not a limit ordinal, then  $L(\xi) = \langle S_\alpha(L(\xi - 1)) \rangle$ . It suffices to see that  $\gamma_1 = \gamma_2$  on the set  $S_\alpha(L(\xi - 1))$ . Let  $b \in S_\alpha(L(\xi - 1))$ . Then there is a set  $A \subset L(\xi - 1)$  with  $|A| < \alpha$  such that  $b = \bigvee^{(L)_\alpha} A$ . So  $\gamma_1(b) = \bigvee^N \gamma_1[A] = \bigvee^N \gamma_2[A] = \gamma_2(b)$ .

(c) Apply 3.2 and (b) above.

(d) Clearly,  $|S_\alpha(L)| < |L|^\alpha$ , so it follows that  $|\langle S_\alpha(L) \rangle| < |L|^\alpha$ . Transfinite induction gives us that for each  $\xi < \alpha$ , we have  $|L(\xi)| < |L|^\alpha$ , hence  $|\langle L \rangle_\alpha| = |\bigcup_{\xi < \alpha} L(\xi)| \leq |L|^\alpha$ .

(e) Clearly, if  $M$  is  $\alpha$ -full in  $N$  and  $L \subseteq M$ , then  $\langle S_\alpha^N(L) \rangle^N \subseteq M$ . Transfinite induction implies that  $L(\xi)^N \subseteq M$  for all  $\xi < \alpha$ , hence  $\langle L \rangle_\alpha^N \subseteq M$ .

(f) Let  $A \subset \langle L \rangle_\alpha$  with  $|A| < \alpha$ . For each  $a \in A$ ,  $a \in L(\xi_a)$  for some  $\xi_a < \alpha$ . Therefore,  $\sup_A \xi_a = \kappa < \alpha$ . It follows that  $A \subset L(\kappa)$ . So if  $b = \bigvee^M A$ , then  $b \in S_\alpha(L(\kappa + 1)) \subset \langle L \rangle_\alpha$ .

To see that  $L$   $\alpha$ -generates  $\langle L \rangle_\alpha$  let  $L \subseteq L' \subseteq \langle L \rangle_\alpha$  and suppose  $L'$  is  $\alpha$ -full in  $\langle L \rangle_\alpha$ . Then  $L'$  is  $\alpha$ -full in  $M$  ((c) here). Therefore it follows that  $L' = \langle L \rangle_\alpha$  ((e) here).

(g) Apply (f) here then 3.4.

**3.8 Definition.** For a given  $L$ ,  $(\varphi, M)$  is called a *free  $\alpha$ -regular extension* of  $L$  if it satisfies the following conditions.

- (i)  $\varphi, M$  are  $\alpha$ -complete.
- (ii)  $\varphi[L]$   $\alpha$ -generates  $M$ .
- (iii) For each  $\alpha$ -complete  $\gamma : L \rightarrow N$  with  $N$   $\alpha$ -complete, there is a unique  $\alpha$ -complete morphism  $\tau : M \rightarrow N$  such that  $\gamma = \tau \circ \varphi$ .

We say that  $(\varphi, M)$  is a *free  $\alpha$ -extension* of  $L$  if it satisfies the following conditions.

- (i')  $M$  is  $\alpha$ -complete.
- (ii')  $\varphi[L]$   $\alpha$ -generates  $M$ .
- (iii') For each  $\gamma : L \rightarrow N$  with  $N$   $\alpha$ -complete, there is a unique  $\alpha$ -complete morphism  $\tau : M \rightarrow N$  such that  $\gamma = \tau \circ \varphi$ .

Note, conditions (iii) and (iii') respectively imply that free  $\alpha$ -regular extensions and free  $\alpha$ -extensions of  $L$  are essentially unique.

Below we construct, for each  $L$ , the free  $\alpha$ -regular extension and the free  $\alpha$ -extension.

For a fixed  $\alpha < \infty$  and  $L$ , let  $T$  be a fixed set with  $|T| = |L|^\alpha$ . Let  $A_\alpha(T)$  be the set of all  $\alpha$ -complete Arch objects which have  $T$  as the underlying set. Let  $I$  and  $J$  be sets of morphisms defined as follows:

$$I = \{ \varphi : L \rightarrow M : M \in A_\alpha(T), \varphi \text{ is } \alpha\text{-complete} \},$$

$$J = \{ \gamma : L \rightarrow M : M \in A_\alpha(T) \}.$$

If we let  $M_\varphi$  and  $M_\gamma$  be the codomains of  $\varphi$  and  $\gamma$  respectively, we can see that  $L$  is naturally embedded in each of the products  $\prod_I M_\varphi$  and  $\prod_J M_\gamma$  by means of the evaluation map, e.g.,  $e : L \hookrightarrow \prod_I M_\varphi$  defined by  $e(a)_\varphi = \varphi(a)$  for all  $a \in L$ . Since the operations in Arch products are coordinatewise it is straightforward to see that  $\prod_I M_\varphi$  and  $\prod_J M_\gamma$  are  $\alpha$ -complete,  $L \subseteq^\alpha \prod_I M_\varphi$ , and  $L \subseteq \prod_J M_\gamma$ . Define

$$FR_\alpha L = \langle L \rangle_\alpha \subseteq^\alpha \prod_I M_\varphi \quad \text{and} \quad F_\alpha L = \langle L \rangle_\alpha \subseteq \prod_J M_\gamma.$$

Note that the products in Arch and  $\mathcal{W}$  are the set-theoretic products with coordinatewise operations, and the product in  $\mathcal{S}$ , of a set of  $\mathcal{S}$ -objects  $M_i$ , is obtained by first forming the product in Arch, and then taking the principal ideal generated by the element  $(u_i)$ , where  $u_i$  is the strong unit of  $M_i$ .

**3.9 Theorem.** Let  $i$  be the inclusion of  $L$  in  $FR_\alpha L$  and  $i'$  be the inclusion of  $L$  in  $F_\alpha L$ . Then  $(i, FR_\alpha L)$  and  $(i', F_\alpha L)$  are the free  $\alpha$ -regular extension and the free  $\alpha$ -extension of  $L$  respectively.

*Proof.* We prove that  $(i, FR_\alpha L)$  is the free  $\alpha$ -regular extension of  $L$ . The proof that  $(i', F_\alpha L)$  is the free  $\alpha$ -extension of  $L$  is similar.

Since  $FR_\alpha L$  is  $\alpha$ -complete,  $F_\alpha L \subseteq^\alpha \prod_I M_\varphi$ , and  $L$   $\alpha$ -generates  $FR_\alpha L$  (3.7(g), (f)), it suffices to see that any  $\alpha$ -complete morphism  $\varphi : L \rightarrow M$ , with  $M$   $\alpha$ -complete, can be extended to an  $\alpha$ -complete morphism  $\bar{\varphi} : \prod_I M_\varphi \rightarrow$

$M$ . We will show that  $\bar{\varphi}$  is essentially a projection out of  $\prod_I M_\varphi$ . (Note, projections are  $\infty$ -complete.)

Let  $\varphi : L \rightarrow M$  be  $\alpha$ -complete and let  $M_{\varphi'} = \langle \varphi[L] \rangle_\alpha^M$ . Since  $M_{\varphi'}$  is  $\alpha$ -complete and  $|M_{\varphi'}| < |L|^\alpha$  (3.7(g), (d)),  $M_{\varphi'}$  is isomorphic to some member of  $A_\alpha(T)$ . Therefore we can consider the morphism  $\varphi' : L \rightarrow M_{\varphi'}$ , where  $\varphi'(a) = \varphi(a)$  for  $a \in L$ , to be a member of  $I$  (3.2). Moreover, since  $M_{\varphi'} \subseteq^\alpha M$  (3.7(g)), we need only to extend  $\varphi$  to an  $\alpha$ -complete morphism  $\bar{\varphi} : \prod_I M_\varphi \rightarrow M_{\varphi'}$ . This is easily done by taking  $\bar{\varphi} = \pi_{\varphi'}$ , where  $\pi_{\varphi'}$  is the  $\varphi'$  projection out of the product  $\prod_I M_\varphi$ . See Figure 3.1.

$$\begin{array}{ccc}
 & \prod_I M_\varphi & \\
 & \alpha \cup | & \\
 & \pi_{\varphi'} & \\
 & FR_\alpha L & \\
 \uparrow & & \\
 \varphi : L & \longrightarrow & M_{\varphi'} = \langle \varphi[L] \rangle_\alpha^M \subseteq^\alpha M
 \end{array}$$

FIGURE 3.1

Recall in an abstract category  $\mathcal{B}$ , a full subcategory  $\mathcal{A}$  is called an *epireflective* subcategory of  $\mathcal{B}$  if, for each  $B \in |\mathcal{B}|$ , there is an  $A_B \in |\mathcal{A}|$  and an epimorphism in  $\mathcal{B}$ ,  $e : B \rightarrow A_B$ , such that for each  $\mathcal{B}$  morphism,  $f : B \rightarrow A$ , to an  $A \in |\mathcal{A}|$ , there exists a (necessarily) unique  $\mathcal{A}$  morphism,  $\bar{f} : A_B \rightarrow A$ , satisfying  $f = \bar{f} \circ e$ .  $(e, A_B)$  is called the  $\mathcal{A}$  *epireflection* of  $B$ . (Note, epireflections are essentially unique.)

The existence of the free  $\alpha$ -regular extension and 3.7(c) together imply:

**3.10 Theorem.** *In  $\text{Arch}(\alpha)$ , full subcategory of  $\alpha$ -complete objects is epireflective, and, for each  $L$ ,  $(i, FR_\alpha L)$  is the  $\alpha$ -complete epireflection of  $L$  in  $\text{Arch}(\alpha)$ .*

Note all the results (in, and about,  $\text{Arch}$  and  $\text{Arch}(\alpha)$ ) of this section have analogs in  $\mathcal{W}$ ,  $\mathcal{W}(\alpha)$ ,  $\mathcal{S}$ ,  $\mathcal{S}(\alpha)$ ,  $\mathcal{BA}$  and,  $\mathcal{BA}(\alpha)$ . Moreover, the proofs, are essentially identical to those in  $\text{Arch}$  and  $\text{Arch}(\alpha)$ . Most importantly, for each  $\mathcal{W}$  (and  $\mathcal{S}$ ) object, there is, in  $\mathcal{W}$  (and  $\mathcal{S}$ ), a free  $\alpha$ -regular extension and a free  $\alpha$ -extension. And, as in  $\text{Arch}(\alpha)$ , we have the following:

**3.11 Theorem.** *In  $\mathcal{W}(\alpha)$  [and  $\mathcal{S}(\alpha)$ ], the full subcategory of  $\alpha$ -complete objects is epireflective, and, for each  $L \in |\mathcal{W}|$  [ $L \in |\mathcal{S}|$ ], the  $\mathcal{W}$  [ $\mathcal{S}$ ] free  $\alpha$ -regular extension  $(i, \mathcal{W} - FR_\alpha L)$  [( $i, \mathcal{S} - FR_\alpha L$ )] is the  $\alpha$ -complete epireflection of  $L$  in  $\mathcal{W}(\alpha)$  [ $\mathcal{S}(\alpha)$ ].*

#### 4. $\alpha$ -DISCONNECTED $\alpha$ -SpFi COREFLECTIONS

Consider the  $\mathcal{W}$ -free  $\alpha$ -regular extension  $(i, FR_\alpha C(X))$ , and the  $\mathcal{W}$ -free  $\alpha$ -extension  $(i', F_\alpha C(X))$ , of  $C(X)$ . Let  $m_\alpha X = Y(FR_\alpha C(X))$  and  $M_\alpha X =$

$Y(F_\alpha C(X))$ .  $(m_\alpha X, Y(i))$  and  $(M_\alpha X, Y(i'))$  are preimages of  $X$ . See Figures 4.1 and 4.2.

$$\begin{array}{ccc} C(X) & \xhookrightarrow{\quad i \quad} & FR_\alpha C(X) \\ X = Y(C(X)) & \xleftarrow{Y(i)} & Y(FR_\alpha C(X)) = m_\alpha X \end{array}$$

FIGURE 4.1

$$\begin{array}{ccc} C(X) & \xhookrightarrow{\quad i' \quad} & F_\alpha C(X) \\ X = Y(C(X)) & \xleftarrow{Y(i')} & Y(F_\alpha C(X)) = M_\alpha X \end{array}$$

FIGURE 4.2

Let  $m_\alpha = Y(i)$  and  $M_\alpha = Y(i')$ . We have the following:

**4.1 Theorem.** *Let  $\alpha < \infty$ .*

- (a)  $m_\alpha X$  and  $M_\alpha X$  are  $\alpha$ -disconnected.
- (b) In  $\alpha$ -SpFi,  $m_\alpha$  is monic.
- (c) If  $f : Y \rightarrow X$  is an  $\alpha$ -SpFi and  $Y$  is  $\alpha$ -disconnected, then there is a unique  $\alpha$ -SpFi morphism  $\bar{f} : Y \rightarrow m_\alpha X$  such that  $f = m_\alpha \circ \bar{f}$ .
- (d) If  $h : Y \rightarrow X$  is continuous and  $Y$  is  $\alpha$ -disconnected, then there is a unique  $\alpha$ -SpFi morphism  $\bar{h} : Y \rightarrow M_\alpha X$  such that  $h = M_\alpha \circ \bar{h}$ .

*Proof.* (a)  $FR_\alpha C(X)$  and  $F_\alpha C(X)$  are  $\alpha$ -complete so  $m_\alpha X$  and  $M_\alpha X$  are  $\alpha$ -disconnected (2.8).

(b)  $i$  is epic in  $\mathcal{W}(\alpha)$ ; thus  $m_\alpha$  is monic in  $\alpha$ -SpFi (2.7).

(c)  $f' : C(X) \rightarrow C(Y)$  is an  $\alpha$ -SpFi morphism (2.4 and 2.6). And since  $C(Y)$  is  $\alpha$ -complete (2.9), there is a unique  $\alpha$ -complete  $\mathcal{W}$  morphism  $\bar{f}' : FR_\alpha C(X) \rightarrow C(Y)$  such that  $f' = \bar{f}' \circ i$ . Hence  $f = Y(f') = Y(\bar{f}' \circ i) = Y(i) \circ Y(\bar{f}') = m_\alpha \circ Y(\bar{f}')$ . Take  $\bar{f} = Y(\bar{f}')$ .  $\bar{f}$  is an  $\alpha$ -SpFi morphism (2.6).

(d) The proof of (d) is similar to that of (c).

Let  $\mathcal{B}$  be a category and  $\mathcal{A}$  a subcategory of  $\mathcal{B}$ . We call  $\mathcal{A}$  a monoreflective subcategory of  $\mathcal{B}$  if, for each  $B \in |\mathcal{B}|$ , there is a  $A_B \in |\mathcal{A}|$  and a monic in  $\mathcal{B}$ ,  $m_B : A_B \rightarrow B$ , such that for each  $\mathcal{B}$  morphism,  $f : A \rightarrow B$ , from an  $A \in |\mathcal{A}|$ , there is a (necessarily) unique  $\mathcal{A}$  morphism,  $\bar{f} : A \rightarrow A_B$ , satisfying  $f = m_B \circ \bar{f}$ .  $(A_B, m_B)$  is called an  $\mathcal{A}$  monoreflection of  $B$ . (Note, monoreflections are essentially unique.)

**4.2 Theorem.** *In  $\alpha$ -SpFi, the full subcategory of  $\alpha$ -disconnected spaces is monoreflective, and for each  $X$ ,  $(m_\alpha X, m_\alpha)$  is the  $\alpha$ -disconnected monoreflection of  $X$  in  $\alpha$ -SpFi.*

*Proof.* Apply 4.1(b) and (c).

**4.3 Remark.** Let  $\text{Ba}(X)$  be the  $\sigma$ -algebra of Baire sets of  $X$  and let  $\mathcal{Z}(X)$  be the  $\sigma$ -ideal generated by the nowhere dense zero-sets of  $X$ . In  $[\text{BH}_3]$  it is shown that  $M_{\omega_1} X$  is the Stone space of  $\text{Ba}(X)$ , and  $m_{\omega_1} X$  is the Stone space of the quotient  $\text{Ba}(X)/\mathcal{Z}(X)$ . Also, for  $X$ , the Stone space of the Borel sets modulo the meager Borel sets is called the absolute or Gleason cover,  $EX$ , of  $X$ .

[PW], and  $m_\infty X \cong EX$  [BHM, W]. See also  $[M_1, M_2]$ . Can  $M_\alpha X$  and  $m_\alpha X$ , for general  $\alpha$ , be represented in a similar fashion? I do not know the answer for arbitrary  $X$ , however, if  $X$  is  $\alpha$ -cozero complemented, the answer is "yes" for  $m_\alpha X$   $[M_3]$ .  $X$  is  $\alpha$ -cozero complemented if, for each  $U \in \text{Coz}_\alpha(X)$ , there is a  $V \in \text{Coz}_\alpha X$  such that  $U \cup V$  is dense in  $X$ , and  $U \cap V = \emptyset$ .

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